

Breaking a chaotic image encryption algorithm based on modulo addition and XOR operation

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This paper re-evaluates the security of a chaotic image encryption algorithm called MCKBA/HCKBA and finds that it can be broken efficiently with two known plain-images and the corresponding cipher-images. In addition, it is reported that a previously proposed breaking on MCKBA/HCKBA can be further improved by reducing the number of chosen plain-images to two from four. The two attacks are both based on some properties of solving a composite function involving carry bit, which is composed of modulo addition and bitwise OR operations. Both rigorous theoretical analysis and detailed experimental results are provided to support the found points.

Keywords: image encryption; chaos; differential attack; carry bit.

1. Introduction

The subtle similarities between chaos and cryptography make chaos considered as a special way to design secure and efficient encryption schemes [Chen *et al.*, 2004, 2011]. Meanwhile, some cryptanalysis work demonstrated that some chaos-based encryption schemes are vulnerable to various conventional attacks from the viewpoint of modern cryptology [Li *et al.*, 2004; Xiao *et al.*, 2006; Solak *et al.*, 2010a,b]. In addition, some specific security flaws of chaos-based encryption schemes were reported [Zhou & Au, 2011; Chen *et al.*, 2012]. [Álvarez & Li, 2006] concluded some general approaches to evaluating security of chaos-based encryption schemes.

Due to the simplicity and low computation complexity of bitwise exclusive OR operation and modulo addition, they are widely used in traditional text encryption schemes and hash functions. Possible generation of carry bit by the modulo addition makes the two operations are neither identical nor interchangeable. Some properties existing in multi-round combination of the two basic operations were derived to facilitate differential attacks on some traditional text encryption schemes or searching collision of hash functions [Paul & Preneel, 2005; Wang *et al.*, 2005]. Among many chaos-based encryption schemes, the two operations are the basic involved (even only) substitution functions. In [Li *et al.*, 2005], [Li *et al.*, 2006], [Li

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et al., 2008], and [Li *et al.*, 2009], the following properties about n -bit integers $\alpha, \beta, \gamma, x, y$ were found to support or enhance the proposed attacks on the corresponding encryption schemes in turn.

- If $\alpha \oplus \beta = 2^n - 1$, then equation $(\alpha \oplus x) = (\beta \oplus x) \dot{+} \gamma$ has unique solution modulo 2^{n-1} , where $(a \dot{+} b) = (a + b) \bmod 2^n$;
- Equation $|(\alpha \oplus \beta) - (\beta \oplus \gamma)| = |(\alpha \oplus \bar{\beta}) - (\beta \oplus \bar{\gamma})|$ always exists;
- If $\alpha \oplus \beta = \gamma$, then $|\alpha - \beta| \leq \gamma$;
- If $((x \dot{+} \alpha) \oplus \beta) \dot{+} \gamma \equiv x \oplus y$, then $y \equiv \beta \pmod{2^{n-1}}$.

In 2000, Yen *et al.* proposed a chaotic key-based algorithm (CKBA) by encrypting each pixel of a plain-image by four possible operations: XORing or XNORing it with one of two predefined sub-keys. The exerted operation is determined by a pseudo-random number sequence (PRNS) generated by iterating the logistic map [Yen & Guo, 2000]. In 2002, S. Li *et al.* broke CKBA with only one known/chosen-image in [Li & Zheng, 2002]. In 2005, Socek *et al.* proposed an enhanced version of CKBA (ECKBA) employing the following four methods: 1) replacing the logistic map with a piecewise linear chaotic map (PWLCM); 2) increasing the bit length of secret key to 128; 3) adding a modulo addition and an XOR operation; 4) running all the basic encryption functions multiple times. To achieve a much better balance between encryption load and security of high level, in 2007 Rao *et al.* proposed a modified version of CKBA (MCKBA) in [Rao & Gangadhar, 2007] by employing a modular addition operation like [Socek *et al.*, 2005]. To further enhance the security of MCKBA against brute-force attack, in 2010 Gangadhar *et al.* replaces the logistic map with a simple hyperchaos generator proposed in [Takahashi *et al.*, 2004] and names the algorithm HCKBA (Hyper Chaotic-Key Based Algorithm) [Gangadhar & Rao, 2010]. Since the two schemes MCKBA and HCKBA share the same structure, [Li *et al.*, 2011] analyzed them together and reported the following points:

- Equivalent secret key of MCKBA/HCKBA can be obtained from four pairs of chosen-plaintexts;
- Encryption result of MCKBA/HCKBA is not sensitive to changes of plain-image;
- Encryption result of MCKBA is not sensitive to changes of two sub-keys.
- The lower bound on the number of queries (α, β) to solve unknown variable x in equation

$$y = (\alpha \dot{+} x) \oplus (\beta \dot{+} x) \quad (1)$$

in terms of modulo 2^{n-1} is 3 if $n \geq 4$.

This paper re-evaluates the security of MCKBA/HCKBA and reports the following points: 1) some properties of Eq. (1) are provided to support practical approaches to solving Eq. (1); 2) MCKBA/HCKBA can be efficiently broken with two known-plaintexts; 3) the chosen-plaintext attack proposed in [Li *et al.*, 2011] can be further improved and the number of required chosen-plaintexts is only two.

The rest of this paper is organized as follows. The image encryption algorithm under study is briefly introduced in Sec. 2. A known-plaintext attack and an improved chosen-plaintext attack on the algorithm is presented in Sec. 3 with experimental results. The last section concludes this paper.

2. The Chaotic Image Encryption Algorithm Under Study

The encryption object of MCKBA is a gray-scale image of size $M \times N$ (width \times height), which is scanned in the raster order and represented as a one dimensional sequence $\mathbf{I} = \{I(i)\}_{i=0}^{MN-1}$. Then, a binary sequence $\mathbf{I}_b = \{I_b(l)\}_{l=0}^{8MN-1}$ is constructed, where $\sum_{j=0}^7 I_b(8 \cdot i + j) \cdot 2^j = I(i)$ for $i = 0 \sim MN - 1$. With a pre-defined integer parameter n , an n -bit number sequence $\mathbf{J} = \{J(k)\}_{k=0}^{\lceil 8MN/n \rceil - 1}$ is generated, where $J(k) = \sum_{j=0}^{n-1} I_b(n \cdot k + j) \cdot 2^j$. In case $(8MN)$ is not a multiple of n , the sequence \mathbf{I}_b is padded with some zero bits. Without loss of generality, it is assumed that n can divide $(8MN)$ in this paper. MCKBA operates on the intermediate sequence \mathbf{J} and obtains $\mathbf{J}' = \{J'(k)\}_{k=0}^{8MN/n-1}$, where $J'(k) = \sum_{j=0}^{n-1} I'_b(n \cdot k + j) \cdot 2^j$. Finally, cipher-image $\mathbf{I}' = \{I'(k)\}_{k=0}^{MN-1}$ is obtained via $I'(k) = \sum_{j=0}^7 I'_b(8 \cdot k + j) \cdot 2^j$. Based on the above

preliminary introduction, MCKBA is described with the following four parts¹.

- *The secret key*: Two random numbers $key_1, key_2 \in \{0, \dots, 2^n - 1\}$, and the initial condition $x(0) \in (0, 1)$ of the logistic map

$$x(k+1) = 3.9 \cdot x(k) \cdot (1 - x(k)), \quad (2)$$

where $\sum_{j=0}^{n-1} (key_{1,j} \oplus key_{2,j}) = \lceil n/2 \rceil$, $key_1 = \sum_{j=0}^{n-1} key_{1,j} \cdot 2^j$, $key_2 = \sum_{j=0}^{n-1} key_{2,j} \cdot 2^j$, and \oplus denotes the eXclusive OR (XOR) operation.

- *Initialization*: Run Eq. (2) iteratively to generate a sequence $\{x(k)\}_{k=0}^{MN/(2^n)-1}$ and derive a pseudo-random binary sequence (PRBS), $\{b(l)\}_{l=0}^{16MN/n-1}$, from the 32-bit binary representation of elements of the sequence, namely $x(k) = \sum_{j=1}^{32} b(32 \cdot k + j - 1) \cdot 2^{-j}$.

- *Encryption*: For $k = 0 \sim 8MN/n - 1$, encrypt the k -th plain-element of \mathbf{J} via

$$J'(k) = \begin{cases} (J(k) \dot{+} key_1) \oplus key_1 & \text{if } B(k) = 3; \\ (J(k) \dot{+} key_1) \odot key_1 & \text{if } B(k) = 2; \\ (J(k) \dot{+} key_2) \oplus key_2 & \text{if } B(k) = 1; \\ (J(k) \dot{+} key_2) \odot key_2 & \text{if } B(k) = 0, \end{cases} \quad (3)$$

where $B(k) = 2 \cdot b(2k) + b(2k+1)$, and $a \odot b = \overline{a \oplus b} = a \oplus \bar{b}$.

- *Decryption*: The decryption procedure is similar to that of the encryption except that Eq. (3) is replaced by

$$J(k) = \begin{cases} (J'(k) \oplus key_1) \dot{-} key_1 & \text{if } B(k) = 3; \\ (J'(k) \oplus \bar{key}_1) \dot{-} key_1 & \text{if } B(k) = 2; \\ (J'(k) \oplus key_2) \dot{-} key_2 & \text{if } B(k) = 1; \\ (J'(k) \oplus \bar{key}_2) \dot{-} key_2 & \text{if } B(k) = 0, \end{cases} \quad (4)$$

where $a \dot{-} b = (a - b + 2^n) \bmod 2^n$.

3. Cryptanalysis

Assume that two plain-images and the corresponding cipher-images encrypted with the same secret key are available, and let $\mathbf{J}_1 = \{J_1(k)\}_{k=0}^{8MN/n-1}$ and $\mathbf{J}_2 = \{J_2(k)\}_{k=0}^{8MN/n-1}$ denote the corresponding intermediate sequences, respectively. Then, one can assure that the two sequences and the corresponding encrypted results $\mathbf{J}'_1 = \{J'_1(k)\}_{k=0}^{8MN/n-1}$ and $\mathbf{J}'_2 = \{J'_2(k)\}_{k=0}^{8MN/n-1}$ satisfy

$$J'_1(k) \oplus J'_2(k) = \begin{cases} (J_1(k) \dot{+} key_1) \oplus (J_2(k) \dot{+} key_1) & \text{if } B(k) \in \{2, 3\}; \\ (J_1(k) \dot{+} key_2) \oplus (J_2(k) \dot{+} key_2) & \text{if } B(k) \in \{0, 1\}. \end{cases} \quad (5)$$

No matter what the value of $B(k)$ is, the above equation can be represented in the form of Eq. (1). In this section, we first present some properties of the kernel function (1) on obtaining its solution and then illustrate how to obtain an equivalent secret key of MCKBA/HCKBA with two known plain-images and two chosen plain-images, respectively.

3.1. Some properties of the kernel function

Property 1. Equivalent form of Eq. (1)

$$\tilde{y} = y \oplus \alpha \oplus \beta = (\alpha \dot{+} x) \oplus (\beta \dot{+} x) \oplus \alpha \oplus \beta \quad (6)$$

¹Since the sole difference between MCKBA and HCKBA is the generator of PRBS, only MCKBA is introduced here with a concise and consistent form to illustrate the encryption procedure.

can be represented as an iteration form

$$\begin{cases} \tilde{y}_{i+1} = c_{i+1} \oplus \tilde{c}_{i+1}, \\ c_{i+1} = (x_i \cdot \alpha_i) \oplus (x_i \cdot c_i) \oplus (\alpha_i \cdot c_i), \\ \tilde{c}_{i+1} = (x_i \cdot \beta_i) \oplus (x_i \cdot \tilde{c}_i) \oplus (\beta_i \cdot \tilde{c}_i), \end{cases} \quad (7)$$

where $c_0 \equiv 0$, $\tilde{c}_0 \equiv 0$, $x = \sum_{i=0}^{n-1} x_i \cdot 2^i$, $\alpha = \sum_{i=0}^{n-1} \alpha_i \cdot 2^i$, $\beta = \sum_{i=0}^{n-1} \beta_i \cdot 2^i$, $\tilde{y} = \sum_{i=0}^{n-1} \tilde{y}_i \cdot 2^i$ (These notations are the same hereinafter.).

Proof. Let c_{i+1} denote the carry bit generated by x and α in the i -th bit plane. Set $c_0 = 0$, we has c_{i+1} from c_i and α_i via

$$c_{i+1} = (x_i \cdot \alpha_i) \oplus (x_i \cdot c_i) \oplus (\alpha_i \cdot c_i) \quad (8)$$

for $i = 0 \sim n - 2$. Similarly, let \tilde{c}_{i+1} denote the carry bit generated by x and β in the i -th bit plane. Set $\tilde{c}_0 = 0$, we can then obtain

$$\tilde{c}_{i+1} = (x_i \cdot \beta_i) \oplus (x_i \cdot \tilde{c}_i) \oplus (\beta_i \cdot \tilde{c}_i)$$

for $i = 0 \sim n - 2$. Obviously, $\tilde{y}_0 = (\alpha_0 \oplus x_0) \oplus (\beta_0 \oplus x_0) \oplus \alpha_0 \oplus \beta_0 \equiv 0$. Then, the $(i + 1)$ -th bit plane of Eq. (6) can be represented as

$$\begin{aligned} \tilde{y}_{i+1} &= (\alpha_{i+1} \oplus c_{i+1} \oplus x_{i+1}) \oplus (\beta_{i+1} \oplus \tilde{c}_{i+1} \oplus x_{i+1}) \oplus \alpha_{i+1} \oplus \beta_{i+1} \\ &= c_{i+1} \oplus \tilde{c}_{i+1}, \end{aligned}$$

where $i = 0 \sim n - 2$. So, \tilde{y}_i can be easily calculated iteratively according to Eq. (7) for $i = 0 \sim n - 2$, which can also be done via checking Table 1 listing the values of \tilde{y}_{i+1} under all possible different values of $\alpha_i, \beta_i, \tilde{y}_i, x_i$, and c_i . ■

Table 1. The values of \tilde{y}_{i+1} corresponding to the values of $\alpha_i, \beta_i, \tilde{y}_i, x_i$, and c_i .

(x_i, c_i)	$(\alpha_i, \beta_i, \tilde{y}_i)$							
	(0, 0, 0)	(0, 0, 1)	(0, 1, 0)	(0, 1, 1)	(1, 0, 0)	(1, 0, 1)	(1, 1, 0)	(1, 1, 1)
(0, 0)	0	0	0	1	0	0	0	1
(0, 1)	0	0	1	0	1	1	0	1
(1, 0)	0	1	1	1	1	0	0	0
(1, 1)	0	1	0	0	0	1	0	0

Property 2. Given $(\alpha_i, \beta_i, \tilde{y}_i, \tilde{y}_{i+1})$, no information about x_i, c_i and \tilde{c}_i can be obtained (Note that c_0 and \tilde{c}_0 are excluded since they are pre-defined constants.) if and only if $(4\alpha_i + 2\beta_i + \tilde{y}_i) \in \{0, 6\}$.

Proof. Since only the data in the 0, 6-th column (zero-based) of Table 1 are identical, it is impossible to obtain any information about x_i, c_i and \tilde{c}_i from $(\alpha_i, \beta_i, \tilde{y}_i, \tilde{y}_{i+1})$ if and only if $(4\alpha_i + 2\beta_i + \tilde{y}_i) \in \{0, 6\}$. ■

Property 3. Given $(\alpha_i, \beta_i, \tilde{y}_i, \tilde{y}_{i+1})$, the unknown bit x_i can be determined via $x_i = \alpha_i \oplus \tilde{y}_{i+1}$, if and only if $(4\alpha_i + 2\beta_i + \tilde{y}_i) \in \{1, 7\}$.

Proof. Only under the cases shown in the 1, 7-th column of Table 1, x_i can be determined by $(\alpha_i, \beta_i, \tilde{y}_i, \tilde{y}_{i+1})$ without knowledge of c_i . It is easy to verify that $x_i = \alpha_i \oplus \tilde{y}_{i+1}$ in terms of value. ■

Property 4. Given $(\alpha_i, \beta_i, \tilde{y}_i, \tilde{y}_{i+1})$, carry bits c_i and \tilde{c}_i can be determined via $c_i = \beta_i \oplus \tilde{y}_{i+1}$ and $\tilde{c}_i = \beta_i \oplus \tilde{y}_{i+1} \oplus \tilde{y}_i$ if and only if $(4\alpha_i + 2\beta_i + \tilde{y}_i) \in \{3, 5\}$.

Proof. Only under the cases shown in the 3, 5-th column of Table 1, c_i can be determined by $(\alpha_i, \beta_i, \tilde{y}_i, \tilde{y}_{i+1})$ without knowledge of x_i . It is easy to verify that $c_i = \beta_i \oplus \tilde{y}_{i+1}$ in terms of value. Then, one can obtain $\tilde{c}_i = \beta_i \oplus \tilde{y}_{i+1} \oplus \tilde{y}_i$ since $\tilde{c}_i = c_i \oplus \tilde{y}_i$. ■

Property 5. Given $(\alpha_i, \beta_i, \tilde{y}_i, \tilde{y}_{i+1})$, the scope of the unknown bits x_i, c_i can be narrowed via

$$(x_i, c_i) \in \begin{cases} \{(0, 0), (1, 1)\} & \text{if } \tilde{y}_{i+1} = 0; \\ \{(0, 1), (1, 0)\} & \text{if } \tilde{y}_{i+1} = 1, \end{cases} \quad (9)$$

if and only if $(4\alpha_i + 2\beta_i + \tilde{y}_i) \in \{2, 4\}$.

Proof. Referring to the cases shown in the 2, 4-th column of Table 1, the scope of (x_i, c_i) can be narrowed according to value of \tilde{y}_{i+1} . It is easy to obtain Eq. (9) from Table 1. Therefore, the “if” part of the property is proven. Note that the number of possible values of $(4\alpha_i + 2\beta_i + \tilde{y}_i)$ is only eight, and the sufficient and necessary conditions on obtaining different information on x_i and c_i under other six cases have been presented. Therefore, the “only if” part of the property is also proven. ■

3.2. Known-plaintext attack

Known-plaintext attack is one of the classic attack models where the attacker (or cryptanalyst) can access both some plaintext and the corresponding encryption results encrypted with the same secret key. In [Gangadhar & Rao, 2010, Sec. 3.2], the original authors claimed that HCKBA has strong vulnerability against known-plaintext attack. However, we found MCKBA/HCKBA is very weak against the attack, which is supported by the properties of Eq. (1) shown in the previous subsection.

Under the scenario of known-plaintext attack, breaking MCKBA/HCKBA is to determine its equivalent secret key, $key1$, $key2$ and $\{B(k)\}_{k=0}^{8MN/n-1}$, by solving Eq. (5) and utilizing some properties of MCKBA/HCKBA. From Proposition 1, one can see that some bits of $key1$ or $key2$ can be obtained from Eq. (5) for any $k \in \{0, \dots, 8MN/n - 1\}$, where the other unknown bits are just set as zero. Let $key(k)$ denote the obtained solution of Eq. (5) and $s(k, i)$ represent $key(k)_i$ is confirmed definitely or not, i.e. set $s(k, i) = 1$ if $key(k)_i$ is confirmed by Eq. (10), Eq. (11), or Eq. (13), otherwise set $key(k)_i = 0$, where $key(k) = \sum_{i=0}^{n-1} key(k)_i \cdot 2^i$, and $k = 0 \sim 8MN/n - 1$. Then, one may reconstruct set $\{key1, key2\}$ from $\{key(k)\}_{k=0}^{8MN/n-1}$ and $\{s(k, i)\}_{k=0, i=0}^{8MN/n-1, n-2}$ by identifying and combining the known bits belonging to the same number, which is described by the following steps.

- *Step 1*): Set $\mathbb{K} = \{key(0), key(1), \dots, key(8MN/n - 1)\}$.
- *Step 2*): Search for two elements in \mathbb{K} whose number of confirmed bits are most but the confirmed bits of the two elements are not all the same. Let $Seed(0)$ and $Seed(1)$ denote the two seed elements and delete them from \mathbb{K} .
- *Step 3*): Check each element of \mathbb{K} in turn and do the following two operations if it has one confirmed bit which is different from that of $Seed(i)$: 1) update $Seed(1 - i)$ by combining all the confirmed bits of the element into that of $Seed(1 - i)$; 2) delete the element from \mathbb{K} , where $i \in \{0, 1\}$.
- *Step 4*): Repeat *Step 3*) iteratively till the numbers of confirmed bits of $Seed(0)$ and $Seed(1)$ are not increased in the whole step.
- *Step 5*): Terminate the whole search operation when all bits of $Seed(0)$ and $Seed(1)$ are confirmed bits.
- *Step 6*): Repeat *Step 2*) through *Step 5*) till the cardinality of \mathbb{K} is less than 2.

Proposition 1. Given α, β, \tilde{y} , the bits among the $(n - 1)$ least significant bits of x in Eq. (6), whose change can cause inexistence of Eq. (6), can be determined from the least significant bit to the most significant one.

Proof. The concrete approaches to solving Eq. (1) and determining the carry bits can be divided into the following two classes of operations.

- *Obtaining information on x_0 and c_1* : According to how much information on x_0 and c_1 can be obtained, $(\alpha_0, \beta_0, \tilde{y}_0)$ is classified as the following two cases.
 - (a) $(4\alpha_0 + 2\beta_0 + \tilde{y}_0) \in \{0, 6\}$: Referring to Property 2, x_0 can not be determined in this case, but one can obtain $c_1 = 0$ if $\alpha_0 = 0$.

169 (b) $(4\alpha_0 + 2\beta_0 + \tilde{y}_0) \in \{2, 4\}$: As $c_0 = 0$, one can obtain

$$x_0 = \begin{cases} 0 & \text{if } \tilde{y}_1 = 0; \\ 1 & \text{if } \tilde{y}_1 = 1, \end{cases} \quad (10)$$

from Eq. (9). Then, one can further obtain

$$c_1 = \begin{cases} 0 & \text{if } \tilde{y}_1 = 0; \\ 1 & \text{if } \alpha_0 = 1 \text{ and } \tilde{y}_1 = 1; \\ 0 & \text{if } \beta_0 = 1 \text{ and } \tilde{y}_1 = 1. \end{cases}$$

170 • *Obtaining information on x_i , x_{i-1} , c_i and c_{i+1} for $i = 1 \sim n - 2$* : According to how much information
 171 on x_i , x_{i-1} , c_i and c_{i+1} can be obtained by checking $(\alpha_i, \beta_i, \tilde{y}_i)$ and the obtained information on c_i for
 172 $i = 1 \sim n - 2$ in order, which is categorized as the following four cases².

- 173 (a) $(4\alpha_i + 2\beta_i + \tilde{y}_i) \in \{0, 6\}$: Referring to Property 2, no information on x can be determined in this case.
 174 The value of c_{i+1} can be determined by Eq. (8) if $(\{c_i + \alpha_i\}, \{\tilde{c}_i + \beta_i\}) \cap \{0, 2\} \neq \emptyset$ is known.
 175 (b) $(4\alpha_i + 2\beta_i + \tilde{y}_i) \in \{1, 7\}$: One has

$$x_i = \alpha_i \oplus \tilde{y}_{i+1} \quad (11)$$

176 from Property 3. If c_i has been determined, one can obtain c_{i+1} . Even c_i is still unknown, one can
 177 confirm c_{i+1} by Eq. (8) if $(\alpha_i + x_i) = 0$ or $(\alpha_i + x_i) = 2$ is known.

- 178 (c) $(4\alpha_i + 2\beta_i + \tilde{y}_i) \in \{2, 4\}$: If c_i has been determined, based on Property 5 one can obtain

$$x_i = \begin{cases} 1 - \tilde{y}_{i+1} & \text{if } c_i = 1; \\ \tilde{y}_{i+1} & \text{if } c_i = 0, \end{cases} \quad (12)$$

179 and further confirm the value of c_{i+1} .

- 180 (d) $(4\alpha_i + 2\beta_i + \tilde{y}_i) \in \{3, 5\}$: Referring to Property 4, one can obtain $c_i = \beta_i \oplus \tilde{y}_{i+1}$. If x_{i-1} is still unknown
 181 but c_{i-1} is known, one can obtain

$$x_{i-1} = \begin{cases} 1 & \text{if } c_i = 1 \text{ and } (c_{i-1} + \alpha_{i-1}) = 1 \text{ is known;} \\ 1 & \text{if } \tilde{c}_i = 1 \text{ and } (\tilde{c}_{i-1} + \beta_{i-1}) = 1 \text{ is known;} \\ 0 & \text{if } c_i = 0 \text{ and } (c_{i-1} + \alpha_{i-1}) = 0 \text{ is known;} \\ 0 & \text{if } \tilde{c}_i = 0 \text{ and } (\tilde{c}_{i-1} + \beta_{i-1}) = 0 \text{ is known.} \end{cases} \quad (13)$$

182 ■

Let us study the probability on obtaining x_i and c_i with one pair of α , β and \tilde{y} under assumption that α , β and x distributes over $\{0, \dots, 2^n - 1\}$ uniformly. First, one has $Prob(c_0 = 1) = 0$ and $Prob(c_i = 1) = \frac{3}{4}Prob(c_{i-1} = 1) + \frac{1}{4}Prob(c_{i-1} = 0)$ for $i = 1 \sim n - 1$. Solve the iteration function, one can obtain $Prob(c_i = 1) = \frac{2^i - 1}{2^{i+1}}$. Observe Table 1, one has $Prob(\tilde{y}_0 = 0) = 1$ and

$$\begin{aligned} Prob(\tilde{y}_i = 0) &= Prob(\tilde{y}_{i-1} = 0) \left(Prob(c_{i-1} = 0) \left(\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} \right) + Prob(c_{i-1} = 1) \left(\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 \right) \right) \\ &\quad + Prob(\tilde{y}_{i-1} = 1) \left(Prob(c_{i-1} = 0) \left(\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \right) + Prob(c_{i-1} = 1) \left(\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \right) \right) \\ &= \frac{3}{4}Prob(\tilde{y}_{i-1} = 0) + \frac{1}{2}Prob(\tilde{y}_{i-1} = 1) \end{aligned}$$

183 for $i = 1 \sim n - 1$. Solve the iteration function, one can obtain

$$Prob(\tilde{y}_i = 0) = \frac{2}{3} + \frac{1}{3 \cdot 4^i}. \quad (14)$$

²As confirmation of c_i is equivalent to that of \tilde{c}_i , the latter is not mentioned.

From the proof of Proposition 1, one can first calculate $Prob[c_0] = 1$, $Prob[c_1] = \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$ and

$$\begin{aligned} Prob[c_i] &= \frac{1}{2} Prob(\tilde{y}_{i-1} = 0) Prob[c_{i-1}] \frac{1}{2} + \frac{1}{2} Prob(\tilde{y}_{i-1} = 1) \cdot \left(Prob[c_{i-1}] + (1 - Prob[c_{i-1}]) \cdot \frac{1}{2} \right) \\ &\quad + \frac{1}{2} Prob(\tilde{y}_{i-1} = 0) Prob[c_{i-1}] + \frac{1}{2} Prob(\tilde{y}_{i-1} = 1) \\ &= Prob[c_{i-1}] \left(\frac{1}{2} Prob(\tilde{y}_{i-1} = 0) + \frac{1}{4} \right) + \frac{3}{4} Prob(\tilde{y}_{i-1} = 1) \\ &= Prob[c_{i-1}] \left(\frac{7}{12} + \frac{1}{6 \cdot 4^{i-1}} \right) + \frac{1}{4} - \frac{1}{4^i} \end{aligned} \quad (15)$$

for $i = 2 \sim n - 2$, where $Prob[a]$ denotes the probability that the bit a can be confirmed. Finally, one has $Prob[x_0] = \frac{1}{2}$ and

$$\begin{aligned} Prob[x_i] &= \frac{1}{2} Prob(\tilde{y}_i = 0) + \frac{1}{2} Prob(\tilde{y}_i = 0) Prob[c_i] \\ &\quad + \frac{1}{2} Prob(\tilde{y}_{i+1} = 1) \left(1 - \frac{1}{2} Prob(\tilde{y}_i = 0) - \frac{1}{2} Prob(\tilde{y}_i = 0) \cdot Prob[c_i] \right) Prob[c_i] \frac{1}{2} \end{aligned} \quad (16)$$

for $i = 1 \sim n - 2$. Incorporate Eq. (14) and Eq. (15) into Eq. (16), one can obtain that $Prob[x_0] = \frac{1}{2}$, $Prob[x_1] = 0.68$, $Prob[x_2] = 0.59$, $Prob[x_3] = 0.57$, and $Prob[x_i] \equiv 0.56$ for $i \geq 4$.

Now, one can assure that $key1_i$ and $key2_i$ can not be confirmed definitely with a probability larger than or equal to $(1 - \frac{1}{2})^{n_0} = \frac{1}{2}^{n_0}$ and $\frac{1}{2}^{(8MN/n - n_0)}$ respectively, where n_0 is cardinality of the set $\{k | B(k) \in \{2, 3\}, k = 0 \sim 8MN/n - 1\}$. Therefore, one can conclude that set $\{key1, key2\}$ can be reconstructed in a very high probability. According to the pre-defined condition $key1 \neq key2$, there are only two possible combinations of $key1$ and $key2$. Let $(key1^*, key2^*)$ denote the searched version of $(key1, key2)$. When there exists $i \in \{0, \dots, n - 2\}$ satisfying that $s(k, i) = 1$, one can obtain approximate scope of $B(k)$,

$$\mathbb{B}^*(k) = \begin{cases} \{2, 3\} & \text{if } key(k)_i = key1_i^* \text{ and } key(k)_i \neq key2_i^*; \\ \{0, 1\} & \text{if } key(k)_i = key2_i^* \text{ and } key(k)_i \neq key1_i^*, \end{cases} \quad (17)$$

for $k = 0 \sim 8MN/n - 1$. From Proposition 2 and Eq. (3), one can obtain the scope of $B(k)$,

$$\mathbb{B}(k) = \begin{cases} \{1, 3\} & \text{if } (J'_1(k) \oplus J_1(k)) \bmod 2 = 0; \\ \{0, 2\} & \text{otherwise,} \end{cases} \quad (18)$$

for $k = 0 \sim 8MN/n - 1$. Then, the approximate value of $B(k)$ can be obtained by setting $B^*(k) = \mathbb{B}^*(k) \cap \mathbb{B}(k)$ for $k = 0 \sim 8MN/n - 1$.

Proposition 2. Assume that a and x are both n -bit integers and $n \in \mathbb{Z}^+$, $((a \dot{+} x) \oplus x)$ has the same parity as a and $((a \dot{+} x) \odot x)$ has opposite parity as a .

Proof. Existence of four equations

$$\begin{aligned} ((1 + x_0) \bmod 2) \oplus x_0 &\equiv 1, \\ ((0 + x_0) \bmod 2) \oplus x_0 &\equiv 0, \\ ((1 + x_0) \bmod 2) \odot x_0 &\equiv 0, \\ ((0 + x_0) \bmod 2) \odot x_0 &\equiv 1, \end{aligned}$$

is independent of x_0 , so the proposition is proved. ■

Finally, one can conclude that $(key1^*, key2^*) = \sum_{i=0}^{n-2} key1_i^* \cdot 2^i$, $\sum_{i=0}^{n-2} key2_i^* \cdot 2^i$, and $\{B^*(k)\}_{k=0}^{8MN/n-1}$ can work together as equivalent secret key of MCKBA/HCKBA due to the following two points: 1) $(key1, key2, B(k)) = (a, b, c)$ and $(key1, key2, B(k)) = (b, a, (c + 2) \bmod 4)$ are equivalent for Eq. (4); 2) Proposition 3 illustrates that the unknown most significant bit of $key1^*$ and $key2^*$ has no influence on decryption of MCKBA/HCKBA.

Proposition 3. Assume that a and x are both n -bit integers, $n \in \mathbb{Z}^+$, one has the following two equations

$$\begin{aligned}(a \oplus x) \dot{-} x &= (a \oplus x \oplus 2^{n-1}) \dot{-} (x \oplus 2^{n-1}), \\ (a \oplus \bar{x}) \dot{-} x &= (a \oplus x \oplus 2^{n-1}) \dot{-} (x \oplus 2^{n-1}).\end{aligned}$$

Proof. See the proof of Proposition 1 in [Li et al., 2011]. ■

To verify the real performance of the above analysis, some experiments are carried out on some plain-images of size 512×512 when $n = 32$. When $x_0 = 319684607/2^{32}$, $key_1 = 3835288501$, and $key_2 = 1437224678$. Two known plain-images “Peppers” and “Baboon”, and the corresponding cipher-images are adopted. Equivalent key key_1^* , key_2^* and $\{B^*(k)\}_{k=0}^{8MN/n-1}$ is used to decrypt another cipher-image shown in Fig. 1a) and the recovered result is shown in Fig. 1b).

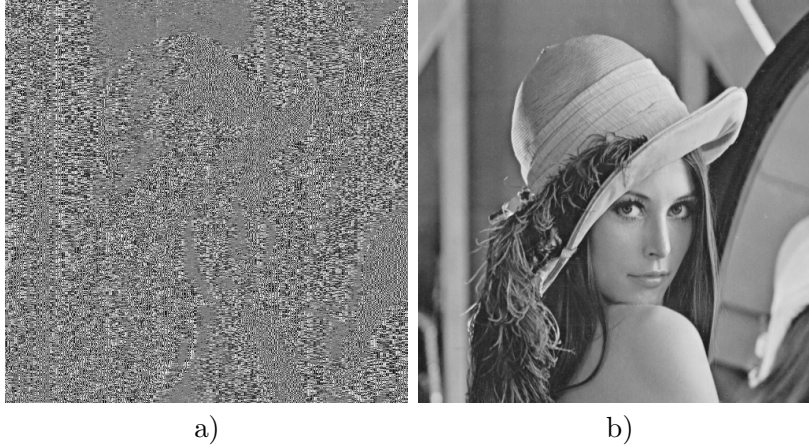


Fig. 1. The decryption result of another cipher-image encrypted with the same secret key: a) cipher-image; b) decrypted plain-image.

3.3. Chosen-plaintext attack

Chosen-plaintext attack is an enhanced version of known-plaintext attack, where the plaintext can be chosen arbitrarily to obtain the information about the secret key in a more efficient way. In this subsection, the chosen-plaintext attack on MCKBA/HCKBA is briefly introduced due to the following two points: 1) the known-plaintext attack on MCKBA/HCKBA works well in a relatively high probability and the chosen-plaintext version can improve its performance a little; 2) the underlying theorem supporting the attack proposed in [Li et al., 2011, Theorem 1] is not right and corrected in Proposition 4.

Proposition 4. Assume that α, β, x are all n -bit integers, then a lower bound on the number of queries (α, β) to solve Eq. (1) in terms of modulo 2^{n-1} for any x is 1 if $n = 2$; 2 if $n > 2$.

Proof. When $n = 2$, one can obtain $x_0 = \tilde{y}_1$ by choosing $(\alpha_0, \beta_0) = (1, 0)$. When $n > 2$, \tilde{y}_1 may be equal to zero or one no matter what (α_0, β_0) is, which means that it is impossible to satisfy the condition of Property 3 for any x . So, we have to resort to another query (α', β') . Let $\alpha'_i, \beta'_i, y'_i, \tilde{y}'_i$ and c'_i denote the counterparts of $\alpha_i, \beta_i, y_i, \tilde{y}_i$, and c_i corresponding to (α', β') . Given a set of $(\alpha_{i+k}, \beta_{i+k})$ and $(\alpha'_{i+k}, \beta'_{i+k})$, one can obtain $(c_{i+k+1}, \tilde{y}_{i+k+1})$ and $(c'_{i+k+1}, \tilde{y}'_{i+k+1})$ from $(c_{i+k}, \tilde{y}_{i+k})$ and $(c'_{i+k}, \tilde{y}'_{i+k})$, respectively, where i, k are non-negative integers. Let arrows of plain head and “V-back” head denote $x_{i+k} = 0$ and $x_{i+k} = 1$, respectively, Fig. 1 illustrates the mapping relationship between $(c_{i+k}, \tilde{y}_{i+k}, c'_{i+k}, \tilde{y}'_{i+k})$ and $(c_{i+k+1}, \tilde{y}_{i+k+1}, c'_{i+k+1}, \tilde{y}'_{i+k+1})$ for a given $(\alpha_{i+k}, \beta_{i+k}, \alpha'_{i+k}, \beta'_{i+k})$, where $k = 0, 1$. Since $(c_0, \tilde{y}_0, c'_0, \tilde{y}'_0) \equiv (0, 0, 0, 0)$, the dashed arrows in Fig. 2 describe operations of Eq. (1) in the two least significant bit planes

corresponding to two set of (α, β) . Note that the data in the third column is exactly the same as the first one. Therefore, Fig. 2 demonstrates operations of Eq. (1) under all different bit levels if the variable i goes through $3 \cdot t$, where $t = 0 \sim \lfloor n/2 \rfloor$ and $i + k \leq n - 1$. Referring to Fig. 2, it can be easily verified that

$$1 \in \{y_i, y'_i\}$$

is always satisfied, which means that x_i can be derived from Table 1. ■

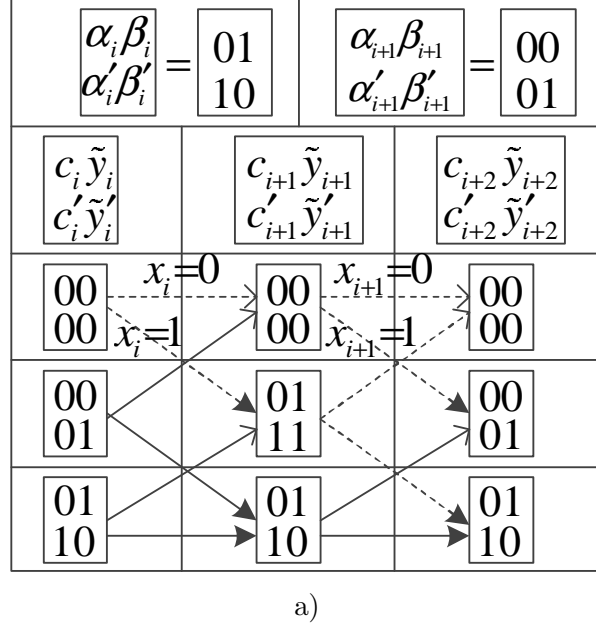


Fig. 2. Relationship between $(c_{i+k}, \tilde{y}_{i+k}, c'_{i+k}, \tilde{y}'_{i+k})$ and $(c_{i+k}, \tilde{y}_{i+k}, c'_{i+k}, \tilde{y}'_{i+k})$ for a given $(\alpha_{i+k}, \beta_{i+k}, \alpha'_{i+k}, \beta'_{i+k})$, where $k = 0, 1$.

Under scenario of chosen-plaintext attack, one may make the plaintext satisfy that at least one pair of elements in $\{(J_1(k), J_2(k)) \mid B(k) \in \{0, 1\}\}$ whose i -th bit plane satisfy the condition of Property 3. The same case exists for $\{(J_1(k), J_2(k)) \mid B(k) \in \{2, 3\}\}$. The expected chosen-plaintext can be obtained in a high probability by assigning $(J_1(k), J_2(k))$ with one of the two sets of number given in Corollary 3.1 randomly. Compared with the known-plaintext attack, the chosen-plaintext attack has the following two superior performances: 1) the set $\{key1, key2\}$ can be reconstructed with much less complexity and much higher degree of accuracy; 2) the bits of $key(k)$ can be confirmed with a little higher probability, where $k = 0 \sim 8MN/n - 1$.

Corollary 3.1. The $(n - 1)$ least significant bits of x in Eq. (1) can be determined easily by setting (α, β) with the following two sets of numbers

$$\left\{ \left(\sum_{j=0}^{\lceil n/2 \rceil - 1} (00)_2 \cdot 4^j \right) \bmod 2^n, \left(\sum_{j=0}^{\lceil n/2 \rceil - 1} (10)_2 \cdot 4^j \right) \bmod 2^n \right\},$$

$$\left\{ \left(\sum_{j=0}^{\lceil n/2 \rceil - 1} (10)_2 \cdot 4^j \right) \bmod 2^n, \left(\sum_{j=0}^{\lceil n/2 \rceil - 1} (01)_2 \cdot 4^j \right) \bmod 2^n \right\},$$

and checking the corresponding $\tilde{y} = y \oplus \alpha \oplus \beta$.

Proof. The proof is straightforward and therefore omitted. ■

4. Conclusion

In this paper, the security of the image encryption algorithm MCKBA/HCKBA has been re-studied in detail. Based on some properties of a composite function composed of modulo addition and XOR operation,

a known-plaintext attack and an improved chosen-plaintext attack were provided to determine an equivalent secret key of MCKBA/HCKBA. The cryptanalysis provided in this paper sheds some light on breaking other encryption schemes based on multiple combination of the modulo addition and XOR operations.

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